UDC 532.5

UNSTEADY FLOW OF INCOMPRESSIBLE FLUID PAST A DOUBLY PERIODIC GRID

PMM Vol. 37, №3, 1973, pp. 488-496

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(Received November 18, 1971)

Validity of the method of superposition of singularities for solving the problem of unsteady flow of incompressible fluid past a doubly periodic grid of arbitrary bodies is proved using the derived below Green's function which has the property of quasi-periodicity. Doubly periodic grids consisting of constant phase shift monopole and dipole elements are examined. Exact integral representation of the perturbation potential of such grids is obtained and its fundamental properties and asymptotic behavior away from the grid are analyzed. According to [1, 2] the solution of the problem of unsteady fluid flow past a grid whose elements oscillate arbitrarily with respect to time τ by the Fourier method reduces to the sum of solutions of the problems of flow past a similar grid whose elements perform oscillations of a single type but in the presence of a constant phase shift between the oscillations of adjacent elements. The problem of three-dimensional flow of incompressible fluid past a grid consisting of bodies with piecewise smooth boundaries, which effect harmonic oscillations is considered below.

Let a doubly periodic grid of period a along the x-axis and b along the y-axis lie in the $z = 0$ plane in the space defined by coordnates xyz. We denote the exterior of the grid by D and its boundary defined as the totality of boundaries of all elements constituing the grid by ∂D, i.e.

$$\partial D = \bigcup_{m, n=-\infty}^{\infty} \Gamma_{mn}, \qquad m, n = 0, \pm 1, \pm 2, \ldots,$$

with $\gamma = \Gamma_{00}$ denoting the boundary of the basic element $(m = 0, n = 0)$ in the rectangular cylinder

$$T = \{x, y, z : |x| < a/2, |y| < b/2, -\infty < z < \infty\}$$

The law defining the oscillations of any element at frequency ω in time τ can be expressed as follows:

$$v_{mn} = v_0 \exp [i(\omega\tau - ma\lambda_1 - nb\lambda_2)]$$

where $v_0 = (v_{0x}, v_{0y}, v_{0z})$ is the vector of the velocity of motion of the basic element, and $a\lambda_1$ and $b\lambda_2$ are oscillation phase shifts of adjacent elements. It is assumed that λ_1 and $\lambda_2 = \text{const}$.

We have to determine the potential $\varphi = \varphi(x, y, z) \in C^2(D) \cup C(\partial D)$ of the incompressible fluid flow past the grid as a solution of the following problem

$$\Delta\varphi = 0, \quad (x, y, z) \in D \tag{1}$$

$$\varphi(x + a, y, z) \exp(i\lambda_1 a) = \varphi(x, y, z) = \varphi(x, y + b, z) \exp(i\lambda_2 b) \tag{2}$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma_{mn}} = \left(v_{0x}\frac{\partial x}{\partial n} + v_{0y}\frac{\partial y}{\partial n} + v_{0z}\frac{\partial z}{\partial n}\right)\exp[-i(ma\lambda_1 + nb\lambda_2)] \tag{3}$$

$$|\varphi| < \infty, \qquad (x, y, z) \in D \tag{4}$$

where (2) is the quasi-periodicity condition, (3) is the bounday condition at the boundary of each element, (4) is the condition of boundedness, and n is the inward normal to surface ∂D. Here and subsequently the time factor $\exp(i\omega\tau)$ is omitted.

Let us prove that function φ can be determined by the method of superposition of singularities, To do this we consider Green's function for the Laplace equation $G = G(x - x_0, y - y_0, z - z_0)$ which has the following properties:

$$G(x + a, y, z) \exp(-i\lambda_1 a) = G(x, y, z) = G(x, y + b, z) \exp(-i\lambda_2 b) \tag{5}$$

$$G = O[\exp(-\alpha|z|)], \quad |z| \to \infty, \quad \alpha \geqslant \alpha_0 > 0 \tag{6}$$

$$\operatorname{Re}\lambda_1 \neq 2\pi m / a, \qquad \operatorname{Re}\lambda_2 \neq 2\pi n / b$$

We denote the region inside the rectangular cylinder T but outside the grid by $T_0 = T \cap D$.

Let $\operatorname{Re}\lambda_1 \neq 2\pi m / a$ and $\operatorname{Re}\lambda_2 \neq 2\pi n / b$. Then, applying to functions φ and G Green's formula, for region T_0 we obtain

$$\iint\limits_{\partial T_0} \left(\varphi \frac{\partial G}{\partial n} - G \frac{\partial \varphi}{\partial n}\right) d\sigma = \iiint\limits_{T_0 \setminus (x_0, y_0, z_0)} (\varphi \Delta G - G \Delta \varphi)\, dx\, dy\, dz$$

Taking into consideration that functions φ and G are quasi-periodic with equal but of opposite sign phase shifts, we can equate to zero the sum of integrals

$$\{x, y, z\colon x = \pm a / 2, \ |y| < b / 2, \ -\infty < z < \infty; \ |x| < a / 2,$$
$$y = \pm b / 2, \ -\infty < z < \infty\}$$

along the side boundary of the rectangular cylinder T_0. The integrals over the infinitely distant edges $|z| = d \to \infty$ tend to vanish by virtue of conditions (4) and (6). Using equalities (1), (3), (4) and (6), we obtain

$$\varphi(x_0, y_0, z_0) = -\frac{1}{4\pi} \iint\limits_{\gamma} \varphi(\xi, \eta, \zeta) \frac{\partial}{\partial n} G(\xi - x_0, \eta - y_0, \zeta - z_0)\, d\sigma +$$
$$\frac{1}{4\pi} \iint\limits_{\gamma} \left(v_{0\xi} \frac{\partial \xi}{\partial n} + v_{0\eta} \frac{\partial \eta}{\partial n} + v_{0\zeta} \frac{\partial \zeta}{\partial n}\right) G(\xi - x_0, \eta - y_0, \zeta - z_0)\, d\sigma$$

We expand the perturbation potential φ into a series in consecutive derivatives of Green's function G similarly to the representation in [2] of the velocity field in a plane in terms of consecutive derivatives of the hyperbolic contangent. For this we represent function G in terms of its Taylor series in the vicinity of point $\xi = \eta = \zeta = 0$

$$G(\xi - x_0, \eta - y_0, \zeta - z_0) = \sum_{q=0}^{\infty} \frac{1}{q!} \left(\xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta}\right)^q G(-x_0, -y_0, -z_0)$$

where q are positive integers. This series is uniformly convergent at any point $(x_0, y_0, z_0) \in T_0$. Owing to this, function φ can be represented by the following expansion in derivatives of G:

$$\varphi(x_0, y_0, z_0) = -\frac{1}{4\pi} \sum_{q=0}^{\infty} \sum_{p=0} \sum_{r=0}^{q-p} \frac{1}{p!\, r!\, (q - p - r)!} \left\{C_{prq}^{\xi} \frac{\partial}{\partial x_0} + C_{prq}^{\eta} \frac{\partial}{\partial y_0} + \right.$$
$$\left. C_{prq}^{\zeta} \frac{\partial}{\partial z_0} - v_{0\xi} M_{prq}^{\xi} - v_{0\eta} M_{prq}^{\eta} - v_{0\zeta} M_{prq}^{\zeta}\right\} \frac{\partial^q G(-x_0, -y_0, -z_0)}{\partial x_0{}^p\, \partial y_0{}^r\, \partial z_0{}^{q-p-r}} \tag{7}$$

where the form coefficients of the considered grid ($\varkappa = \xi, \eta, \zeta$) are

$$C_{prq}^{(\varkappa)} = \iint\limits_{\Upsilon} \varphi(\xi, \eta, \zeta)\, \xi^p \eta^r \zeta^{q-p-r} \frac{\partial \varkappa}{\partial n} d\sigma, \quad M_{prq}^{(\varkappa)} = \iint\limits_{\Upsilon} \xi^p \eta^r \zeta^{q-p-r} \frac{\partial \varkappa}{\partial n} d\sigma$$

It will be shown in the following that function G represents the potential of perturbations generated by a grid of oscillating monopoles. The first derivatives of this function are potentials of perturbations generated by grids of oscillating dipoles oriented parallel to the coordinate axes. The nth order derivatives of function G are potentials for grids of oscillating multipoles of the nth order. The potential of perturbation generated by a doubly periodic grid of oscillating bodies can, thus, be constructed by the method of superposition of singularities by formula (7).

To obtain the explicit form of function G we first consider the doubly periodic grid of oscillating dipoles of intensity varying from one source to another according to the law

$$M_{mn} = abM_0 \exp\left[-i\left(ma\lambda_1 + nb\lambda_2\right)\right], \quad m, n = 0, \pm 1, \pm 2, \ldots$$

with the fundamental dipole ($m, n = 0$) of intensity M_0 lying at the coordinate origin. By directing the axes of all dipoles parallel to the z-axis, function g_1 – the perturbation potential – can be represented in the form of an infinite series

$$g_1 = \frac{ab}{2\pi} M_0 z \sum_{m,n=-\infty}^{\infty} \frac{\exp\left[-i\left(ma\lambda_1 + nb\lambda_2\right)\right]}{\left[(x+am)^2 + (y+bn)^2 + z^2\right]^{3/2}}$$

This series can be summated by the method described in [3]. Using the gamma function representation, we can express the sought series in the form

$$g_1 = \frac{ab}{\pi^{3/2}} M_0 z \int\limits_0^\infty \sum_{m,n=-\infty}^{\infty} \exp\{-\left[(x+am)^2 + (y+bn)^2 + z^2\right] t - i\left(ma\lambda_1 + nb\lambda_2\right)\} \sqrt{t}\, dt$$

In terms of standard notation for the theta function [4] the integrand in the last formula is of the form

$$\vartheta_3\left[iaxt - \lambda_1 a \mid ia^2t/\pi\right] \vartheta_3\left[ibyt - \lambda_2 b \mid ib^2t/\pi\right] \times \exp\left[-(x^2+y^2+z^2)t\right] \sqrt{t}$$

Applying the Jacobi transformation to the theta functions ϑ_3 [4], for the potential of perturbation generated by a doubly periodic grid of dipoles, we finally obtain the integral expression

$$g_1 = \frac{M_0}{\sqrt{\pi}} z \exp\left[i\left(\lambda_1 x + \lambda_2 y\right)\right] \int\limits_0^\infty \vartheta_3\left[\frac{\pi}{a}\left(x + i\frac{\lambda_1}{2t}\right) \middle| \frac{i\pi}{a^2 t}\right] \times \vartheta_3\left[\frac{\pi}{b}\left(y + i\frac{\lambda_2}{2t}\right) \middle| \frac{i\pi}{b^2 t}\right] \exp\left(-\frac{\lambda_1^2 + \lambda_2^2}{4t} - z^2 t\right) t^{-1/2} dt \qquad (8)$$

This formula is convenient for investigations, since the integrand is factorized in it with respect to variables x, y, z and parameters λ_1 and λ_2. Note that for $\lambda_1^2 + \lambda_2^2 = 0$ the equality (8) yields the particular case presented in [3, 5] in which the potential of perturbation induced by a grid of cophased dipoles is expressed by

$$g_1 = M_0 \operatorname{sign} z + \frac{M_0}{\sqrt{\pi}} z \int\limits_0^\infty \left\{\vartheta_3\left[\frac{\pi}{a} x \middle| \frac{i\pi}{a^2 t}\right] \vartheta_3\left[\frac{\pi}{b} y \middle| \frac{i\pi}{b^2 t}\right] - 1\right\} e^{-z^2 t} t^{-1/2} dt$$

From this directly follows the asymptotic estimate for $|z| \to \infty$

$$g_1 = M_0 \operatorname{sign} z + \exp(-\alpha |z|), \quad \alpha = \min(2\pi / a,\ 2\pi / b) \tag{9}$$

The absence of phase shift results in that the perturbation generated by a grid of dipoles extends to infinity.

Let $\lambda_1^2 + \lambda_2^2 \neq 0$ which implies the presence of phase shifts in the oscillations of dipoles. Then, expressing theta functions in the integrand of formula (8) in terms of the Fourier series, we obtain

$$g_1 = \frac{M_0}{\sqrt{\pi}} z \exp[i(\lambda_1 x + \lambda_2 y)] \sum_{m,\, n=-\infty}^{\infty} \exp(2\pi i \tau_{mn}) \int_0^{\infty} \exp\left(-\beta_{nm} \frac{1}{t} - z^2 t\right) t^{-1/2} dt \tag{10}$$

$$\tau_{mn} = (mx / a) + (ny / b), \quad \beta_{mn} = [(\pi m / a) + (\lambda_1 / 2)]^2 + [(\pi n / b) + (\lambda_2 / 2)]^2$$

If λ_1 and λ_2 are multiples of $2\pi/a$ and $2\pi/b$, respectively, we obtain from equality (10) an asymptotic estimate of the kind of (9). If λ_1 and λ_2 are real but not multiples of $2\pi/a$ and $2\pi/b$, the integrals in the last equality are computable and yield the following result:

$$g_1 = M_0 \operatorname{sign} z \exp[i(\lambda_1 x + \lambda_2 y)] \sum_{m,\, n=-\infty}^{\infty} \exp(2\pi i \tau_{mn} - 2|z|\sqrt{\beta_{mn}})$$

We have thus established that the potential of perturbation in the presence of real phase shifts between the oscillations of dipoles, which are not multiples of $2\pi/a$ and $2\pi/b$, is doubly-quasi-periodic and is attenuated at infinity in accordance with the law

$$g_1 = M_0 \operatorname{sign} z \exp[i(\lambda_1 x + \lambda_2 y) - |z| \sqrt{\lambda_1^2 + \lambda_2^2}] + O[\exp(-2|z|\sqrt{\beta_{11}})]$$

A similar estimate can be obtained also for complex λ_1 and λ_2, when $\operatorname{Re} \beta_{mn} > 0$ for any integral m and n, since the character of convergence of the integral in formula (8) is the same

$$g_1 = M_0 \operatorname{sign} z \exp[i(\lambda_1 x + \lambda_2 y) - |z| \sqrt{\lambda_1^2 + \lambda_2^2}] + O[\exp(-2|z| \operatorname{Re} \sqrt{\beta_{11}})], \quad |z| \to \infty$$

The assumption that λ_1 and λ_2 are complex implies that the oscillation intensity of any two dipoles differ not only in phase but, also, in amplitude

$$M_{mn} = A_{mn} \exp[i(am \operatorname{Re} \lambda_1 + nb \operatorname{Re} \lambda_2)],$$
$$A_{mn} = M_0 \exp(-ma \operatorname{Im} \lambda_1 - nb \operatorname{Im} \lambda_2)$$

A different result is obtained in the case of existence of such numbers m_0 and n_0 for which the inequality $\operatorname{Re} \beta_{mn} < 0$ is valid. In fact, the integrals in formula (10) are divergent. To extend these analytically with respect to parameters β_{mn} into the region in which $\operatorname{Re} \beta_{mn} < 0$, it is sufficient to represent function g_1 in the form

$$g_1 = \frac{M_0}{\sqrt{\pi}} z \exp[i(\lambda_1 x + \lambda_2 y)] \int_0^{\infty} \left\{ \vartheta_3\left[\frac{\pi}{a}\left(x + i\frac{\lambda_1}{2t}\right) \middle| \frac{i\pi}{a^2 t}\right] \times \vartheta_3\left[\frac{\pi}{b}\left(y + i\frac{\lambda_2}{2t}\right) \middle| \frac{i\pi}{b^2 t}\right] \exp\left(-\frac{\lambda_1^2 + \lambda_2^2}{4t}\right) - \right.$$

$$\sum_{m_0, n_0} \exp(2\pi i \tau_{mn} - \beta_{mn}\frac{1}{t})\Big\} e^{-z^2 t} t^{-1/2} dt +$$

$$\frac{M_0}{\sqrt{\pi}} z \exp[i(\lambda_1 x + \lambda_2 y)] \sum_{m_0, n_0} \exp(2\pi i \tau_{mn}) \int_0^\infty \exp(-\beta_{mn}\frac{1}{t} - z^2 t)\, t^{-1/2} dt$$

The integrals appearing in the second sum are Bessel functions with index $1/2$ [4]. As the result, we obtain for function g_1 the following analytic extension:

$$g_1 = \frac{M_0}{\sqrt{\pi}} z \exp[i(\lambda_1 x + \lambda_2 y)] \int_0^\infty \Big\{ \vartheta_3\Big[\frac{\pi}{a}\Big(x + i\frac{\lambda_1}{2t}\Big)\Big|\frac{i\pi}{a^2 t}\Big] \times$$
$$\vartheta_3\Big[\frac{\pi}{b}\Big(y + i\frac{\lambda_2}{2t}\Big)\Big|\frac{i\pi}{b^2 t}\Big] \exp\Big(-\frac{\lambda_1^2 + \lambda_2^2}{4t}\Big) -$$
$$\sum_{m_0, n_0} \exp\Big(2\pi i \tau_{mn} - \beta_{mn}\frac{1}{t}\Big)\Big\} e^{-z^2 t} t^{-1/2} dt +$$
$$M_0 \operatorname{sign} z \exp[i(\lambda_1 x + \lambda_2 y)] \sum_{m_0, n_0} \exp(2\pi i \tau_{mn} + i|z|\sqrt{-\beta_{mn}})$$

with the branch of function $\sqrt{-\beta_{mn}}$ chosen so that $\operatorname{Im}\sqrt{-\beta_{mn}} > 0$ for all numbers m_0 and n_0. In the last equality the integral is convergent, which implies that the perturbation potential at considerable distance from the grid ($|z| \to \infty$) tends to vanish

$$g_1 \sim M_0 \operatorname{sign} z \exp[i(\lambda_1 x + \lambda_2 y) + i|z|\sqrt{-\lambda_1^2 - \lambda_2^2}] + O[\exp(-2|z|\sqrt{-\beta_{11}})]$$

Of interest is the case of existence of two such numbers m_0' and n_0' for which the equality $\operatorname{Im}\sqrt{-\beta_{m_0' n_0'}} = 0$ is satisfied. This obtains for $\operatorname{Re}\lambda_1 = -2\pi m_0' / a$ and $\operatorname{Re}\lambda_2 = -2\pi n_0' / b$. In that case the perturbation potential at considerable distance from the grid does not tend to vanish but behaves as a plane wave

$$g_1 \sim M_0 \operatorname{sign} z \exp[-x \operatorname{Im}\lambda_1 - y \operatorname{Im}\lambda_2 + i|z|\sqrt{(\operatorname{Im}\lambda_1)^2 + (\operatorname{Im}\lambda_2)^2}]$$

Thus, if the grid is tuned so that the real parts of dipole phase shifts are multiples of $2\pi/a$ and $2\pi/b$, a plane wave propagates from the grid in a direction determined by the variation of amplitudes of these dipoles.

Let us consider particular cases. Taking into consideration the equality [4]

$$\lim_{b\to 0} \vartheta_3\Big[\frac{\pi}{b}\Big(y + i\frac{\lambda_2}{2t}\Big)\Big|\frac{i\pi}{b^2 t}\Big] = 1$$

and tending period b and phase shift λ_2 to zero in expression (8) for g_1, for the potential of perturbation induced by a one-dimensional grid of oscillating dipoles lying along the x-axis (the plane case) we obtain the expression

$$g_1 = \frac{M_0}{\sqrt{\pi}} z e^{i\lambda x} \int_0^\infty \vartheta_3\Big[\frac{\pi}{a}\Big(x + i\frac{\lambda}{2t}\Big)\Big|\frac{i\pi}{a^2 t}\Big] \exp\Big(-\frac{\lambda^2}{4t} - z^2 t\Big) t^{-1/2} dt, \quad \lambda = \lambda_1$$

Representing the theta function in the last equality in terms of a Fourier series and computing the derived integrals, for real λ we obtain

$$g_1 = M_0 \operatorname{sign} z \exp(i\lambda x) \sum_{m=-\infty}^{\infty} \exp\Big[2\pi i\frac{m}{a}x - \Big|z\Big(2\pi\frac{m}{a} + \lambda\Big)\Big|\Big]$$

If the phase shift is a multiple of $2\pi/a$, for the perturbation potential at a distance from the grid we obtain the asymptotic estimate

$$g_1 \sim M_0 \operatorname{sign} z + O[\exp(-\alpha|z|)], \quad |z| \to \infty$$

which is characteristic of cophase oscillations of dipoles ($\lambda = 0$) .

If λ is small, i.e. $0 < \lambda < 2\pi/a$, we obtain the expression

$$g_1 = (M_0/2) \operatorname{sign} z\{[\operatorname{cth} \pi(|z|-ix)/a + 1] \exp[-\lambda(|z|-ix)] + [\operatorname{cth} \pi(|z|+ix)/a - 1] \exp[\lambda(|z|+ix)]\}$$

whose asymptotic estimate for $|z| \to \infty$ is

$$g_1 \sim M_0 \operatorname{sign} z \exp[-\lambda(|z| - ix)]$$

These results coincide with those derived in [1]. If λ is such that $0 < 2\pi m_0/a < \lambda < 2\pi(m_0+1)/a$, then g_1 can be represented by

$$g_1 = \frac{M_0}{2} \operatorname{sign} z e^{i\lambda x}\Big\{e^{-\lambda|z|} \operatorname{cth}\frac{\pi}{a}(|z| - ix) + e^{\lambda|z|} \operatorname{cth}\frac{\pi}{a}(|z| + ix) - 2\operatorname{sh}\lambda|z| - 4\sum_{m=1}^{m_0} \exp\Big(-2\pi i m\frac{x}{a}\Big)\operatorname{sh}\Big[\Big(\lambda - 2\pi\frac{m}{a}\Big)|z|\Big]\Big\}$$

If $-2\pi(m_0+1)/a < \lambda < -2\pi m_0/a$, then

$$g_1 = \frac{M_0}{2} \operatorname{sign} z e^{i\lambda x}\Big\{e^{-\lambda|z|} \operatorname{cth}\frac{\pi}{a}(|z| - ix) + e^{\lambda|z|} \operatorname{cth}\frac{\pi}{a}(|z| + ix) + 2\operatorname{sh}\lambda|z| + 4\sum_{m=1}^{m_0} \exp\Big(2\pi i m\frac{x}{a}\Big)\operatorname{sh}\Big[\Big(\lambda + 2\pi\frac{m}{a}\Big)|z|\Big]$$

From the last two formulas we obtain the asymptotics of function g_1 for $|z| \to \infty$ and any real phase shift λ

$$g_1 \sim \begin{cases} M_0 \operatorname{sign} z \exp[-\lambda(|z| - ix)], & \lambda > 0 \\ M_0 \operatorname{sign} z \exp[\lambda(|z| + ix)], & \lambda < 0 \end{cases}$$

Similar estimates can be obtained for complex λ. The most interesting case is that of $\operatorname{Re}\lambda = -2\pi m/a$ (m is any arbitrary integer), in which the perturbation potential at a distace from the grid behaves as a plane wave

$$g_1 \sim M_0 \operatorname{sign} z \exp[-x \operatorname{Im}\lambda + i|z||\operatorname{Im}\lambda|], \quad z \to \infty$$

Let us now consider another particular case. If in equality (8) $b \to \infty$ and function g_1 is normalized beforehand with respect to b, then for the potential of the perturbation induced by singly-periodic grid of oscillating dipoles lying in space xyz along the x-axis, we obtain the expression

$$g_1 = \frac{M_0}{\pi} z e^{i\lambda x}\int_0^\infty \vartheta_3\Big[\frac{\pi}{a}\Big(x + i\frac{\lambda}{2t}\Big)\Big|\frac{i\pi}{a^2 t}\Big]\exp\Big(-\frac{\lambda^2}{4t} - \rho^2 t\Big)dt, \quad \rho^2 = y^2 + z^2$$

where the following equality is taken into consideration:

$$\lim_{b\to\infty}\frac{1}{b}\sqrt{\frac{\pi}{t}}\,\vartheta_3\Big[\frac{\pi}{b}\Big(y + i\frac{\lambda_2}{2t}\Big)\Big|\frac{i\pi}{b^2 t}\Big]\exp\Big[\Big(y + i\frac{\lambda_2}{2t}\Big)^2 t\Big] = 1$$

In the considered case for the potential g_1 for any real λ, away from the grid ($\rho \to \infty$)

we have the following asymptotic estimate:

$$g_1 \sim \begin{cases} M_0 z/\pi\rho^2, & \lambda = -2\pi m/a \\ M_0 z\sqrt{|\lambda|\pi\rho^3}\exp(i\lambda x - 2|\lambda|\rho), & \lambda \neq -2\pi m/a \end{cases} \quad (m = 0, \pm 1, \pm 2, \ldots)$$

Thus for a single-row grid in a three-dimensional space the perturbation potential away from the grid tends to vanish for any real λ .

Next, let us consider the doubly-periodic grid of oscillating monopoles with intensity varying according to the law

$$M_{mn} = ab\, M_0 \exp\,[i\,(m\lambda_1 a + n\lambda_2 b)],\ m, n = 0, \pm 1, \pm 2, \ldots$$

The perturbation potential g_0 induced by such grid can be determined by direct integration of expression (8) with respect to z

$$g_0 = \int g_1(x, y, z)\,dz = -\frac{M_0}{2\sqrt{\pi}}\exp\,[i\,(\lambda_1 x + \lambda_2 y)]\int_0^\infty \vartheta_3\left[\frac{\pi}{a}\left(x + i\frac{\lambda_1}{2t}\right)\middle|\frac{i\pi}{a^2 t}\right] \times$$
$$\vartheta_3\left[\frac{\pi}{b}\left(y + i\frac{\lambda_2}{2t}\right)\middle|\frac{i\pi}{b^2 t}\right]\exp\left(-\frac{\lambda_1^2 + \lambda_2^2}{4t} - z^2 t\right)t^{-3/2}\,dt \qquad (11)$$

As in the preceding cases we can establish for the potential g_0 for $|z| \to \infty$ the following asymptotic estimate:

$$g_0 \sim \begin{cases} M_0|z|, \quad \lambda_1 = -2\pi m/a, \quad \lambda_2 = -2\pi n/b \\ -\dfrac{M_0}{\sqrt{\lambda_1^2+\lambda_2^2}}\exp\,[i\,(\lambda_1 x + \lambda_2 y) - |z|\sqrt{\lambda_1^2+\lambda_2^2}], \quad \mathrm{Re}\,\lambda_1 \neq -2\pi m/a, \\ \mathrm{Re}\,\lambda_2 \neq -2\pi n/b \\ M_0 \exp\,[-\mathrm{Im}\,(x\lambda_1 + y\lambda_2) + i|z|\sqrt{(\mathrm{Im}\,\lambda_1)^2 + (\mathrm{Im}\,\lambda_2)^2}] \\ \mathrm{Re}\,\lambda_1 = -2\pi m/a,\ \mathrm{Im}\,\lambda_1 = 0;\ \mathrm{Re}\,\lambda_2 = -2\pi n/b,\ \mathrm{Im}\,\lambda_2 = 0 \end{cases}$$

The sought Green's function $G(x - x_0,\, y - y_0,\, z - z_0)$ is determined by formula (11) and is of the form

$$G(x - x_0, y - y_0, z - z_0) = -\left(\frac{2\pi}{ab}M_0\right)g_0(x - x_0, y - y_0, z - z_0) =$$
$$\frac{\sqrt{\pi}}{ab}\exp\{i\,[\lambda_1(x - x_0) + \lambda_2(y - y_0)]\}\int_0^\infty \vartheta_3\left[\frac{\pi}{a}\left(x - x_0 + i\frac{\lambda_1}{2t}\right)\middle|\frac{i\pi}{a^2 t}\right] \times$$
$$\vartheta_3\left[\frac{\pi}{b}\left(y - y_0 + i\frac{\lambda_2}{2t}\right)\middle|\frac{i\pi}{b^2 t}\right]\exp\left[-\frac{\lambda_1^2 + \lambda_2^2}{4t} - (z - z_0)^2 t\right]t^{-3/2}\,dt$$

It is thus proved that the effective method of superposition of singularities can be used for solving the problem of unsteady flow of an incompressible fluid past a doubly-periodic grid of three-dimensional bodies by constructing doubly-quasi-periodic Green's function for the Laplace equation. The substitution of superposed grids of multipoles of various orders (7) oscillating at constant phase shift for a grid of three-dimensional physical bodies does not affect the action of the latter. The amplitudes of these multipoles are determined by the form coefficients of the considered grid, which depend only on the latter geometric properties.

The properties of Green's functions and of perturbation potentials for grids of oscillating mono- and dipoles have been investigated.

The author thanks M. I. Gurevich for discussing certain results of this investigation and for his valuable advice.

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Translated by J. J. D.

UDC 532.5

THE FORM OF FRESH-WATER LENS FOR LINEAR EVAPORATION LAW

PMM Vol. 37, №3, 1973, pp. 497-504
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(Received October 17, 1972)

Solution of the problem of the stabilized lens of fresh-water filtering from a channel is derived. At the free surface of the latter the stream function is specified in the form of a linear combination of coordinates which includes the particular relationships previously considered by Emikh [1]. The boundary separating fresh and saline waters, the free surface, and the characteristic dimensions of the lens are determined with the use of the analytic theory of linear differential equations.

1. Statement of the problem. The geometry of the considered flow region is shown in Fig. 1. A porous medium of constant porosity m and filtration coefficient K occupies the lower half-plane $y \leqslant 0$. Fresh water of density ρ_1 filters from the channel $A'BA$ of width $2h$, and penetrates the surface of the more sense ground water depressing it in the form of a lens $G \cup G'$. It is assumed that the saline water of density ρ_2 $(\rho_2 > \rho_1)$ lying below the separation

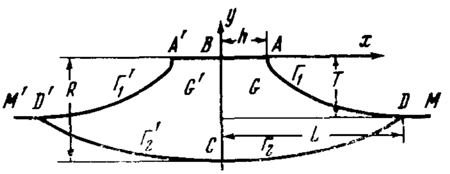

Fig. 1